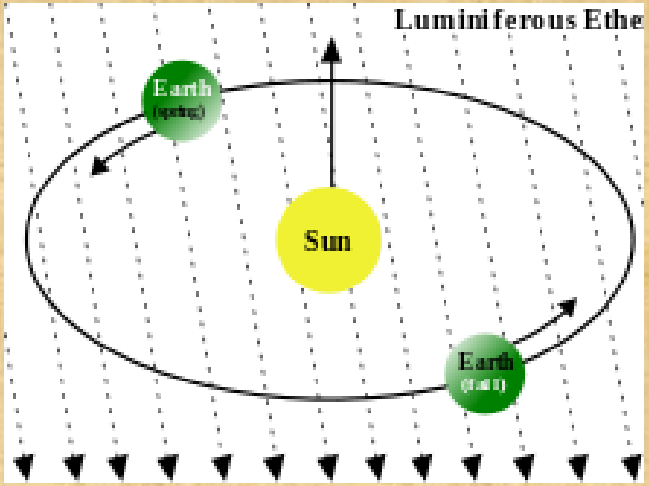
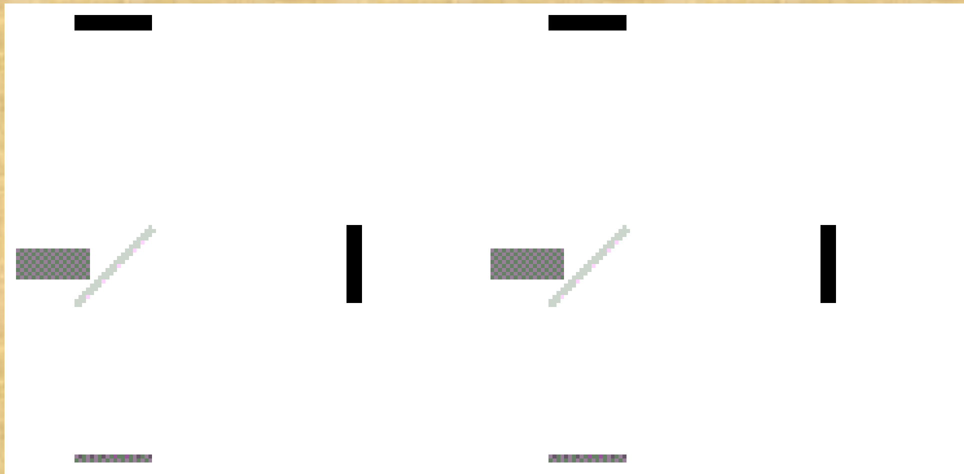
BH Astrophys Ch6.4~6.5

The need to rethink space and time





Any such beam of any kind of particles generated at the speed of light by a moving observer would be received by a stationary observer at that same speed – regardless of how fast the two observers were moving relative to one another.

How could the old Galilean view of space and time work to explain this?

Apparently, it couldn't! We need to change our view of space and time!

Note: the Lorentz transformations came way before this deeper insight of geometry started to take hold.

A note of history

The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality. – Hermann Minkowski, 1908

The line element in spacetime

Previously, in 3D Euclidian geometry, we learned that $\mathrm{dl}^2 = \mathrm{dx}^2 + \mathrm{dy}^2 + \mathrm{dz}^2$ is invariant under coordinate transformations.

Considering space and time together and taking into account that the speed of light must be constant in any frame, we find that the *new* line element to satisfy such conditions is

$$\mathrm{ds}^2 = -c^2\,\mathrm{d}t^2 + \mathrm{dx}^2 + \mathrm{dy}^2 + \mathrm{dz}^2$$

Verification：

Let a photon travel along the x direction. Given that we have two observers (A&B) with different velocities along the x axis,

For A, the line element is then $\mathrm{ds}^2 = -c^2\,\mathrm{d}t'^2 + c^2\mathrm{d}t'^2 = 0$
For B, the line element is still $\mathrm{ds}^2 = -c^2\,\mathrm{d}t^2 + c^2\mathrm{d}t^2 = 0$

This is because both of them see the photon traveling at c ！

An interesting fact about this is that photons apparently travel along what is called null geodesics, the line elements are 0 in any frame ！

A twist in ideas

One other interesting property that follows directly by considering the following line element

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

Is that even when we are apparently sitting still, given that some time dt has elapsed, then we have already traveled by an interval square of $\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2$!

So much for ‘common sense‘ of ‘not moving’ !

Based on the idea of line elements, we must also be able to derive the findings using pure algebra as were first during the development of special relativity.

Namely, we should derive:

1. The effect of Time Dilation
2. The effect of FitGerald Contraction (or Lorentz Contraction)
3. The Lorentz Transforms

Time Dilation

Let's again consider two observers, A&B. To simplify the problem, we only consider x-directional motion.
Let's have A sit still in some jet moving with const. velocity v in the +x direction w.r.t B, then,

For A, she would observe herself only traversing in time, therefore,

$$\mathrm{ds^2}_A = -c^2\mathrm{dt}^2$$

As for B, he would observe the interval as

$$\mathrm{ds^2}_B = -c^2\mathrm{dt}'^2 + v^2\mathrm{dt}'^2$$

Assuming intervals are invariant, (we haven't proven that the invariance of intervals gives the correct Lorentz transforms yet) $\mathrm{ds^2}_A = \mathrm{ds^2}_B$

This then gives

$$\mathrm{dt}' = \frac{\mathrm{dt}}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \equiv \gamma\mathrm{dt}$$

Which is the time dilation formula.

FitGerald Contraction

sure for us v'. Seeing that our clocks tick more slowly, but knowing that our speed is still v', the people in the other spaceship conclude that in our reference frame, our spaceship must have traveled only a distance

$$\Delta\ell = v'\,\Delta t = \frac{v'\,\Delta t'}{\gamma'} = \frac{\Delta\ell'}{\gamma'} \tag{6.45}$$

in the time interval Δt that we measure. So, to the other space travelers, it looks as if our distances are all shorter by a factor of $1/\gamma'$. This is called the FitzGerald contraction, after the Irish physicist George FitzGerald, who proposed this process in 1889. Of course, from our point of view, our distances have not changed.

Some reminders

Postulates of special relativity:

1. The laws of physics are the same in all inertial frames of reference

2. The speed of light in free space has the same value c in all inertial frames.

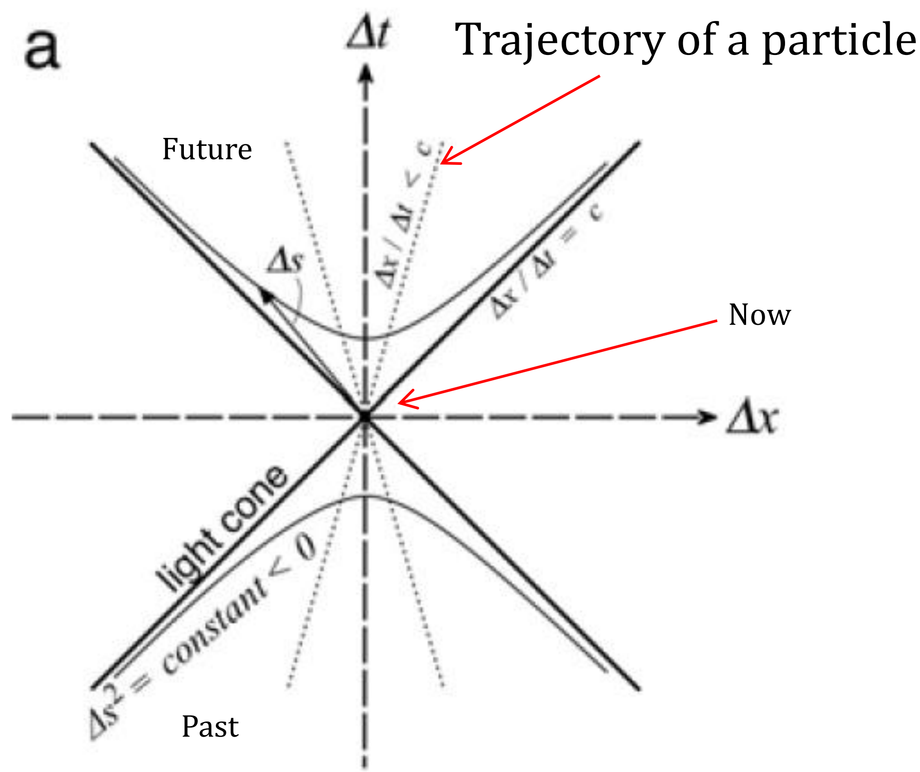
On the Minkowski metric:

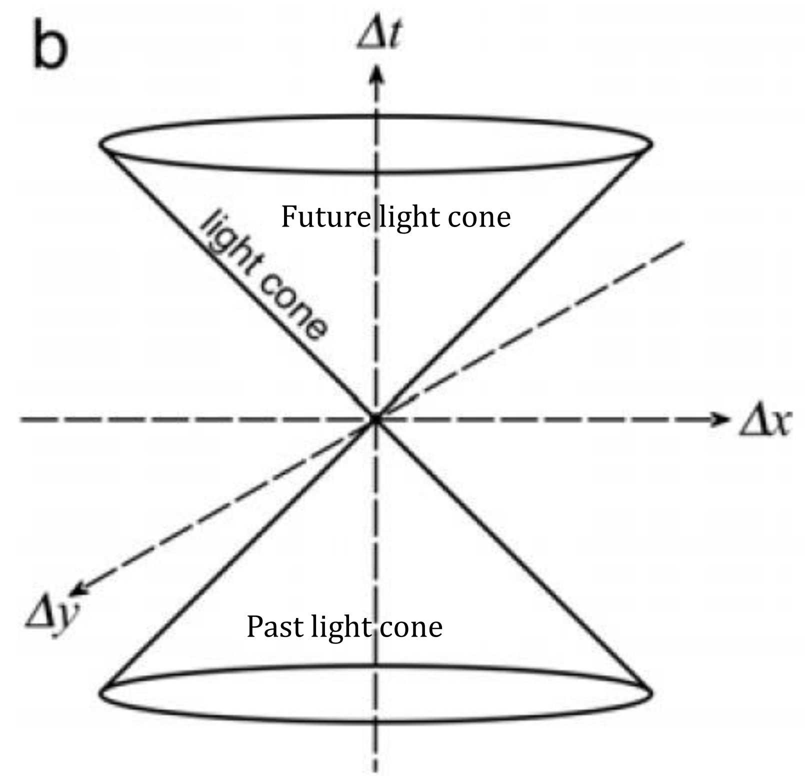
$$(6.41)\ \mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

1. It works for observers moving at any speed up to c.
2. Equation (6.41) is the same for travelers moving at different speeds.
3. It is an extension of the Euclidean metric, although definitely not simply four-dimensional Euclidean.
4. The true fabric of the universe includes both space and time.

Spacetime diagrams

Any event that you can affect lies in your future light cone.





Any event that has the chance of affecting you at this very moment lies in your past light cone.

$ds^2 < 0$ timelike intervals
$ds^2 > 0$ spacelike intervals
$ds^2 = 0$ lightlike intervals

Example

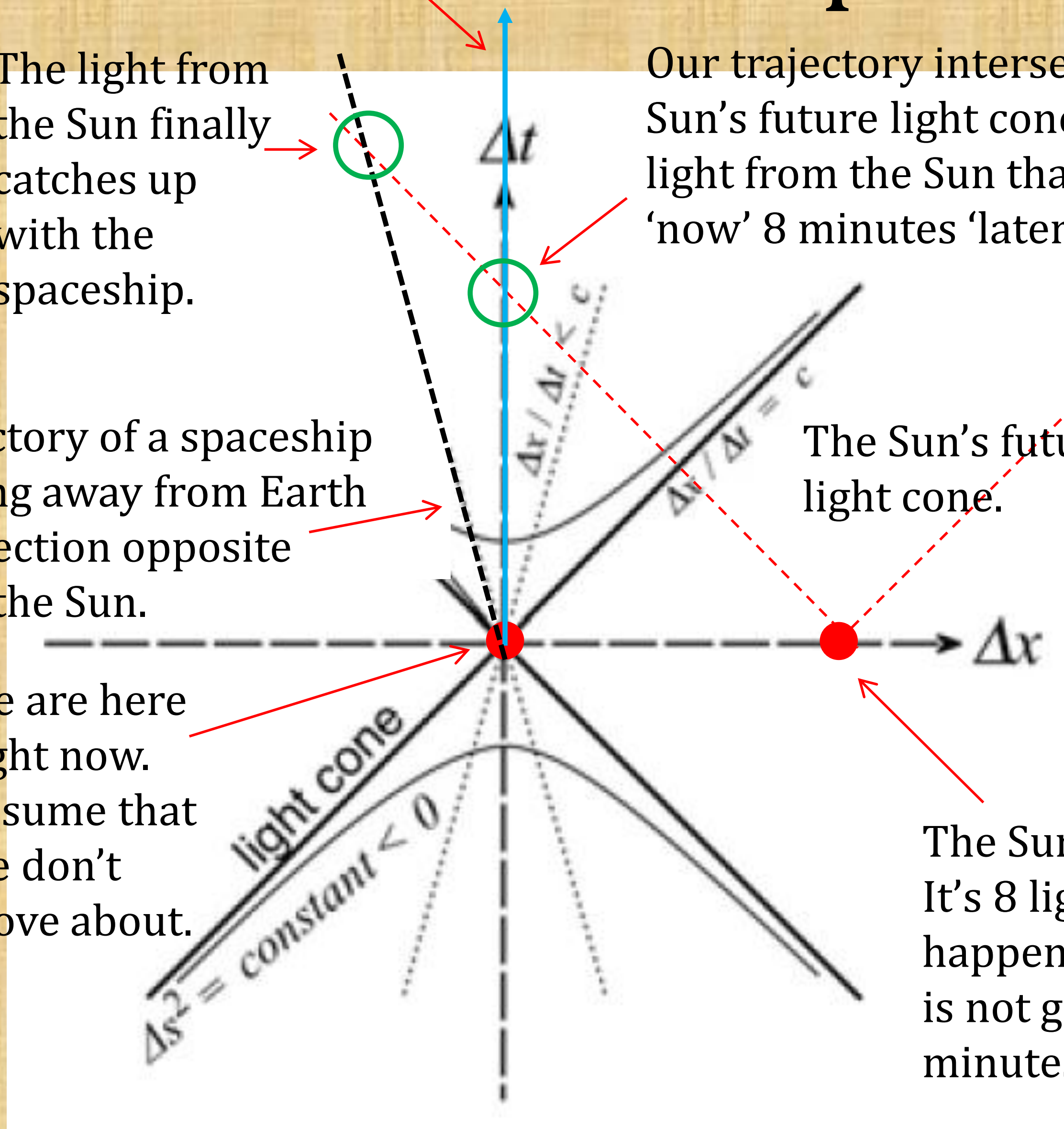
Our trajectory in spacetime.

The light from the Sun finally catches up with the spaceship.

Our trajectory intersects with the Sun's future light cone – we see the light from the Sun that is emitted 'now' 8 minutes 'later.'

Trajectory of a spaceship moving away from Earth in direction opposite from the Sun.

The Sun's future light cone.



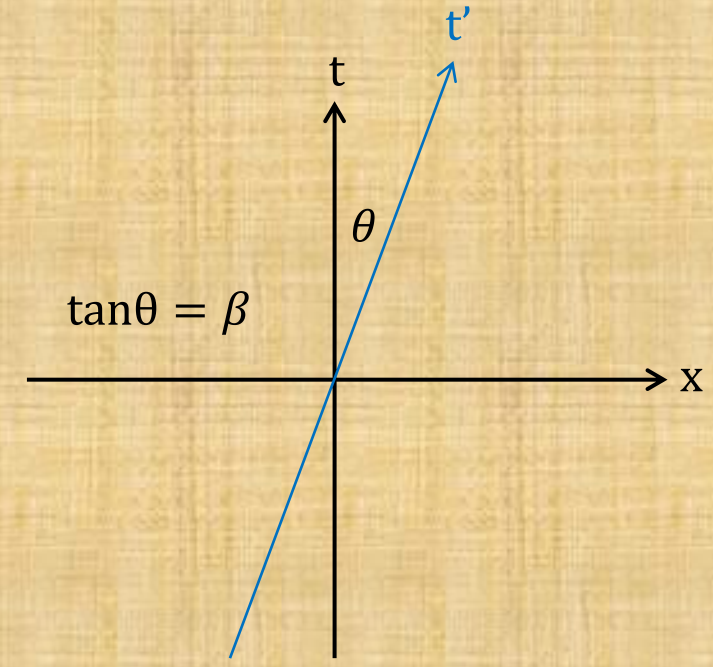
We are here right now. Assume that we don't move about.

The Sun right now.
It's 8 light minutes away, what happens over there right now is not going to affect us till 8 minutes later.

Different inertial frames

To illustrate different observers situated in different inertial frames, it is convenient to overlay spacetime diagrams as seen by two different observers. It will also form the basis of our derivation of Lorentz Transformations using the spacetime diagram.

In the last few slides we just simply drew the spacetime diagram without discussing what exactly we mean by x axis and t axis.



First, the t axis:
As we have shown, when we are not moving, then our path corresponds to the t axis. Or in general, the t axis is the trajectory of the observer which perceives himself as stationary.

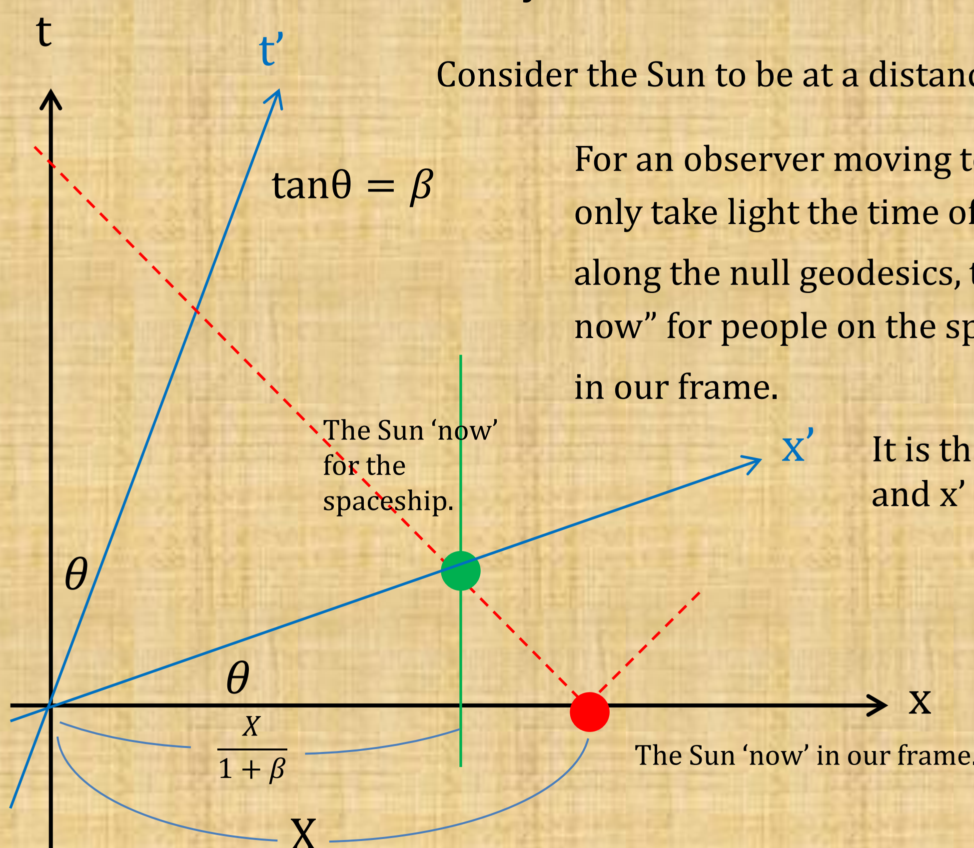
Thus, for an observer that we see a moving in the +x direction, his t' axis would be his trajectory.

Different inertial frames

How about the x axis?

The x axis basically shows all events that happen at the same time. How then do we draw the line that corresponds to events of which in the moving observer's frame are at the same time?

A more formal way can be found in Schutz but here is a very simple illustration:



Consider the Sun to be at a distance x away in our frame.

For an observer moving toward it in a spaceship at velocity β (=v/c), it would only take light the time of $\frac{1}{c}\frac{X}{1+\beta}$ to reach the spaceship. As light always travels along the null geodesics, the point which should be registered as "the Sun right now" for people on the spaceship therefore must be at a distance $\frac{X}{1+\beta}$ as seen in our frame.

It is then easy to work out that the angle between the x and x' axes is also θ by considering the light ray.

One must take caution that the above arguments are done in our frame of reference, people on the spaceship won't think it took time $\frac{1}{c}\frac{X}{1+\beta}$ for the light from the Sun to reach them.

The Lorentz transformations

$t' = \gamma t - \beta\gamma x$ $\qquad$ $x' = -\beta\gamma t + \gamma x$

$$L = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

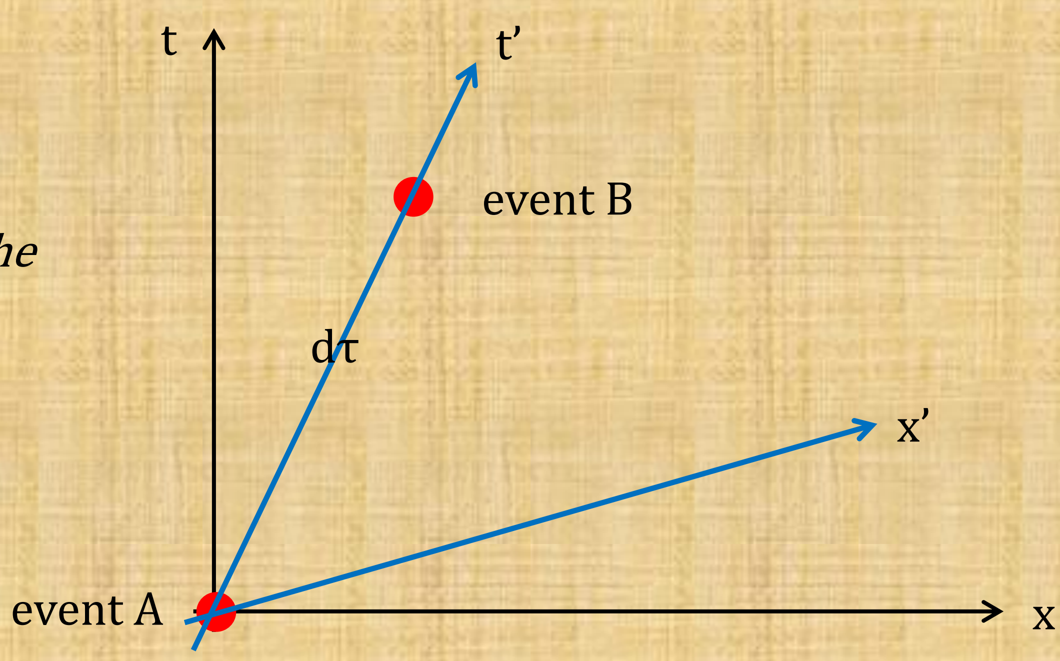
Proper time

In classical physics, we had defined $v = \frac{\mathrm{dx}}{\mathrm{dt}}; a = \frac{\mathrm{dv}}{\mathrm{dt}}$

However, now that time itself isn't some universal quantity. It becomes necessary to discuss what we should use to replace the 'dt' term with.

Apparently, it would be not such a good idea to use dt to define some more universal 'velocity/acceleration' which works in 4 dimensions since dt itself is subjected to change when we go to other frames of reference.

Therefore, we define the quantity called 'Proper time dτ' as :
Given two timelike events, the proper time is the time between the two events as measured by a stationary observer that passes through them.



4-vectors

The Lorentz Transforms were used for transforming the 4-displacement (i.e. coordinates in 4D) in-between different inertial frames of reference.

Therefore, we can define a class of objects called '4-vectors' written as $A^{\mu}$to have the property : 4-vectors follow the same transform as the coordinates transform.

The most basic 4-vector is of course $x^{\mu} = (ct, x, y, z)$. It obviously transforms from one coordinate to another by means of Lorentz Transforms as we've found.

A simple extension would be to define $U^{\alpha} \equiv \frac{\mathrm{d}x^{\alpha}}{\mathrm{d}\tau}$, which we call the '4-velocity' and $a^{\alpha} \equiv \frac{\mathrm{dU}^{\alpha}}{\mathrm{d}\tau}$, which we call '4-acceleration'.

Both of them also transform in-between coordinates like the 4-displacement x^{μ}. This is because we have defined dτ, the proper time to be a scalar quantity, i.e. it is a quantity that doesn't change with coodinates.

Examples

Defining $U^{\alpha} \equiv \frac{\mathrm{dx}^{\alpha}}{\mathrm{d}\tau}$ and $a^{\alpha} \equiv \frac{\mathrm{dU}^{\alpha}}{\mathrm{d}\tau}$, it would be useful to see what they look like in 4-form.

Consider a moving spaceship with const. velocity
Then, for people on it, they would consider themselves as stationary, meaning that their displacement is only $\overline{\mathrm{d}x^{\mu}} = (c\mathrm{d}\tau, 0,0,0)$

*the bar symbol is for moving frame

$$L = \begin{pmatrix} \gamma & +\beta\gamma & 0 & 0 \\ +\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, $\overline{U^{\alpha}} = (c, 0,0,0)$ and $\overline{a^{\alpha}} = (0,0,0,0)$

If we transform $\overline{U^{\alpha}}$ to U^{α} by using $U^{\beta} = \Lambda^{\beta}{}_{\alpha}\overline{U^{\alpha}}$, then we find

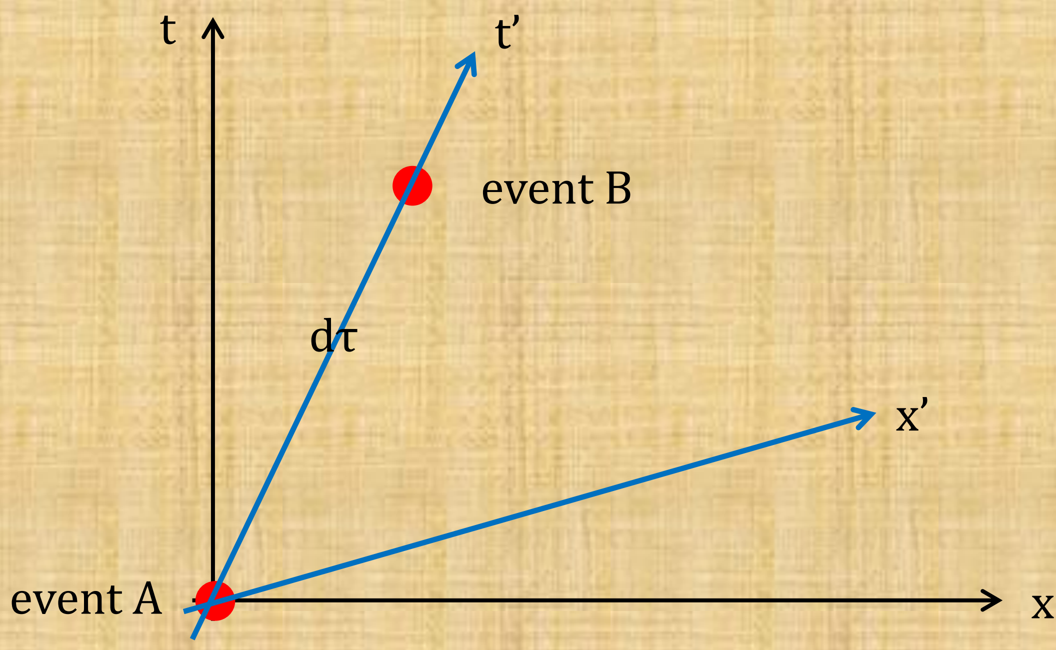
$$U^{\alpha} = (\gamma \mathrm{c}, \gamma v, 0,0)$$

Which looks familiar… except with some extra γ s in there. Where are they from?

Example (cont.)

Remember that $U^\alpha \equiv \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}$ but our classical velocity is $v = \frac{\mathrm{d}x}{\mathrm{d}t}$! So we need to find the relation between dt and dτ. From time-dilation, that would be $\mathrm{d}t = \gamma \mathrm{d}\tau$

Thus, $\frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = U^\alpha \frac{\mathrm{d}\tau}{\mathrm{d}t} = (c, v, 0{,}0)$ which is exactly the trajectory that we drew for a moving spaceship on the spacetime diagram.



The magnitude of 4-velocity

Similar as in 3D case, the magnitude of 4-vectors can be found by $g_{\alpha\beta} U^{\alpha} U^{\beta} = -c^2$

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ is the metric in flat spacetime

Energy-momentum 4-vector

Another quantity that follows quite straight forwardly from the 4-velocity is 4-momentum.

$$P^\alpha \equiv m_0 U^\alpha \,;\, P^2 = -{m_0}^2 c^2$$

m_0 the rest mass of the particle.

Following previous example in which we see a spaceship to have $U^\alpha = (\gamma \mathrm{c}, \gamma v, 0{,}0)$,

$$P^\alpha = (\gamma m_0 c, \gamma m_0 v, 0{,}0)$$

What is the first term in this vector?

By expanding to first order we see that $\gamma m_0 c \approx \frac{1}{c}\left(m_0 c^2 + \frac{1}{2} m_0 v^2\right)$ which simply is the rest mass energy plus the classical kinetic energy!

Therefore, we usually generalize this concept to define $P^\alpha = \left(\frac{E}{c}, P^x, P^y, P^z\right)$

Combining with $P^2 = -{m_0}^2 c^2$, we arrive at the equation $E^2 = P^2 c^2 + {m_0}^2 c^4$

Newton's Law

Similar with what we've done before, Newton's Law $F = \frac{\mathrm{dp}}{\mathrm{dt}}$ in 4-form then is :

$$F^{\mu} = \frac{\mathrm{dP}^{\mu}}{\mathrm{d\tau}}$$

Or in general,

$$F^{\mu} = U^{\mu} P^{\alpha}{}_{,\alpha} \text{ ?}$$

The Doppler factor

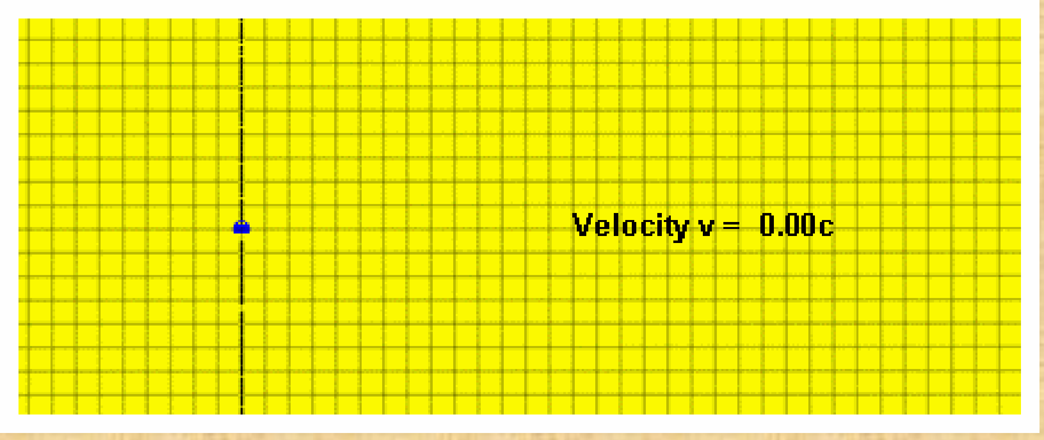
Just as in classical physics, waves exhibit Doppler shifts when viewed from frames with different velocities.

We define the doppler factor δ through $f = \delta f_0$

For relativistic velocities, $\delta = \frac{1}{\gamma(1-\beta\cos\theta)}$

Some cases:

1. Source moving toward observer at v~c $\delta \sim 2\gamma$
2. Source receding from observer at v~c $\delta \sim \frac{1}{2\gamma}$
3. Sources not approaching nor receding $\delta = \frac{1}{\gamma}$ (time dilation)



Importance in astrophysics

The tremendous enhancement in the Doppler effect for relativistic motion is called "Doppler boosting" or "Doppler beaming". It affects many physical quantities (besides just frequency of the emitted light or radio waves) that are measured when astronomers observe relativistic jet flow. Some, such as brightness, depend on the Doppler factor to a high power like 3 or 4. Fig. 6.6 shows a plot of different powers of the Doppler factor as a function of the jet viewing angle θ for $\gamma = 10$. The Doppler factor does not drop formally to a value of 1 until it reaches an angle given by $\cos\theta = [(\gamma - 1)/(\gamma + 1)]^{1/2}$ (or $\theta = 0.440$ radians for $\gamma = 10$). However, we see from the figure that the ef
an exponent of $n = 3$ or 4 is only abou

θ_l

is the effective "beaming angle". For
very near the speed of light with respe
are moving at an angle $\theta > \theta_b$ to our li